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Discrete Topological Space as a Space in Finite Geometry with Variables in Integer Modulom

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ABSTRACT

This paper explores the construction of discrete topological spaces on non-near-linear finite geometries, where points and lines are defined over the ring of integers modulo m. By introducing a partial order among subgeometries, we demonstrate how such structures satisfy the axioms of topology and provide illustrative examples for specific cases such as finite geometry G_6. The geometry under discourse together with the collection of its subsets called subgeometry yields a discrete topological space with its subgeometries as topology.

Keywords: Topology, topological space, discrete topology, ring of integer modulo m, lines, Non-near-linear finite geometry.

INTRODUCTION AND PREAMBLES

Lots of works ongoing focusing on finite geometry, that is a geometry with distinct numbers of points and lines. This is likely due to its application in areas such as computer coding in message transmission, digital security, experimental design etc, for more details see [1]. This growing trend drew our attention to this area of research. In this article, Z_m represents an integer modulo m. The Cartesian products $Z_m \times Z_m$ gives a finite geometry G_m. The geometry is defined as the pair (L_m, P_m) where L_m denotes the set of lines in a geometry and P_m denotes the set of points in a line. where, for instance, two lines in a finite geometry meet at a point. The findings in [2] disagreed with this common axiom by restricting its validity of the axiom to lines in near-linear geometry. The work of [3] further affirmed that such axiom is only satisfied in near-linear finite geometry and that, in non-near-linear finite geometry, two lines of the geometry intersect in at least two points.

In the past, many challenges have always been posited to researchers in the area of topology focusing on the application of topology that is relevance to providing solution to human necessity. For instance, the authors in [4-6] have asked a pertinent question about the areas of application or relevance of topology to humanity outside teaching. In an attempt to respond to those challenge, Mayila et al. [7] developed a mathematical representation of a decision space and a topology on a nation using some properties of topological operators. This further stresses the importance of topology and the need for more studies in order to explore other areas of real-life applications.

In this work, certain terminologies were used, they are discussed thus; the ring of integers modulo m is denoted by Z_m. Finite geometry is denoted by G_m or Z_m^2, so we use the duo interchangeably. Reduced residue integer modulo m, it is represented by Z^*_m. Its cardinality is a Euler Totient function. The lowest common multiplier and greatest common divisor are denoted by LCM and GCD respectively.

We focus on defining a topology on a finite geometry. We examine how topological structure is established on non-near-linear finite geometry. We demonstrate how finite geometry under consideration satisfies conditions of topology. Finally, we demonstrate how divisors function play an essential role in establishing topology on the geometry under discussion. This work is divided as follows. Section I covers an introduction to this work and terminologies used in it. Non-near-linear finite geometry of large dimension is discussed in section II, it is titled, finite geometry $Z_m \times Z_m$. Decomposition of large dimensional finite geometry as products of many small dimensional finite geometries is discussed in III. It is titled, Finite Geometry $Z_m \times Z_m$ as $\prod_{i=1}^{k} Z_{m_i} \times Z_{m_i}$. In section IV, topological space in the context of finite geometry is investigated. In section V, we showcase illustration of this work using numerical examples. Section VI focuses on the conclusion.

FINITE GEOMETRY $Z_m \times Z_m$

In this section, lines in near-linear finite geometry are discussed. This concept forms the theoretical background through which this work is built on. The concepts are defined [1-2] as follows:

Definition II. 1: A space $S(P,L)$ is a system of points P and line L such that every line L is a subset of P.

Definition II. 2: A near linear space is an incident structure $I(P,L)$ of points P and lines L such that the following axioms are satisfied:

Any line has at least two points.
Two lines meet in at most one point.

A near-linear space is defined as follows:

$$G_m = (L_m, P_m) \tag{3}$$

Here, P_m represents points on the line L_m.

L_m denotes lines with point P_m, where

$$L_m = \{(k\mu, k\delta) \mid \mu, \delta \in Z_m, \}, k \in Z_m \tag{4}$$

Lemma II.3: Let m be a prime. Two distinct lines of a near-linear finite geometry G_m intersect at one point.

Proof:

Let $G_m = Z_m^2$

Where $Z_m^2 = Z_m \times Z_m$ represents lines with points in G_m. For m a prime, an intersection of any pair of arbitrary lines yields a point.

Definition II.3: A partial ordered relation R is set Z_m is a relation $R \subseteq Z_m \times Z_m$ which satisfies the following conditions;

i. Reflexivity; that is for $a \in Z_m$, $(a,a) \in Z_m \times Z_m$
ii. Antisymmetric, that is for $a, b \in Z_m$, if $a \prec b$ and $b \prec a$, then $(a,b)=(b,a)$
iii. Transitivity, that is for $a,b,c \in Z_m$, if $(a,b) \prec (b,c)$ and $(b,c) \prec (a,c)$, then $(a,b) \prec (a,c)$.

In this work, $\prec$ represents partial ordering.

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FINITE GEOMETRY $Z_m \times Z_m$ AS $\prod_{i=1}^{k} Z_{m_i} \times Z_{m_i}$

Our attention centres on lines in non-near-linear finite geometry. Specifically, lines in this type of geometry $Z_m \times Z_m$ meet in more than one point. A line through the origin is defined in equation (4).

Our investigation in this work focuses on lines through the origin in the geometry as mentioned in [9]. It is defined as follows:

$$L_m = \{k\mu + \gamma, k\delta + \theta \mid \mu, \delta, \gamma, \theta \in Z_m, \},\ k \in Z_m \quad (5)$$

Mathematically, it is defined as the pair (P_m, L_m) in $G_m = Z_m^2$. Here,

P_m represents points in a line and L_m represents lines in G_m where,

$$P_m = \{(e,f) \mid e, f \in Z_m\} \quad (6)$$

The following propositions were verified and confirmed from related work of [2], [9,10].

1. If k is an invertible element in Z_m, then $L(\mu, \delta) = L(k\mu, k\delta)$
Otherwise $L(\mu,\delta)\ mod(d) \subset L(k\mu, k\delta)$
Hence $L(k\mu, k\delta) \prec L(\mu,\delta)$, where $\prec$ represents partial ordering.
It was established that $L(\mu,\delta)$ is a maximal line in G_m if $GCD(\mu,\delta) \in Z_m^*$ and $L(\mu,\delta)$ is a subline in G_m if $GCD(\mu,\delta) \in Z_m - Z_m^*$
2. Equation (5) can also be defined as
$$L(t\mu, t\delta) = \{(t\rho\mu, t\rho\delta) \mid \mu, \delta \in Z_m\},\ t \in Z_m \quad (7)$$
Furthermore, the line $L(\tau\alpha, \tau\beta)$ in $G_{\tau m}$ is a subline of
$$L(\mu,\delta) = \{(\dot{t}\mu, \dot{t}\delta) \mid \dot{t} = 0, \ldots, \rho m - 1\} \quad (8)$$
3. If n is a divisor of m, then two maximal lines meet at n points, and the n points gives a subline $L(\mu,\delta)$ where $\mu, \delta \in \frac{m}{n} Z_n$.

Considering the subgeometry G_n, the subline $L(\mu,\delta)$ in G_m is a maximal line in G_n. A presence of $\psi(m)$ maximal lines is confirmed in subgeometry G_n of finite geometry G_m. A set of lines in non-near-linear finite geometry G_m together with its set of subgeometry forms a topological space via partial ordering with its subgeometry as embedded in the space.

TOPOLOGICAL SPACE IN $Z_m \times Z_m$ AND $\prod_{i=1}^{k} Z_{m_i} \times Z_{m_i}$

Definition IV.I: Let X be a non-empty set. X is called a discrete topology if every subset of X is an open set.

In this work, we define a discrete topological space over a non-near-linear finite geometry $Z_m \times Z_m$. Lines of the geometry are generated. For m a non-prime, $Z_m \times Z_m$ forms a non-near-linear finite geometry. Properties of topology are tested on the geometry under discourse and the outcome of our investigation confirmed our claim. The concept is investigated thus;

Definition IV.II: A pair $(Z_m, \tau_{Z_m \times Z_m})$ where Z_m denotes a nonempty set Z_m and $\tau_{Z_m \times Z_m}$ denotes collection of subsets is called a topological space if it meets the following conditions:

i. The empty set and the whole set are elements of τ, that is, $\phi, Z_m \in \tau_{Z_m \times Z_m}$
ii. The union of any member of $\tau_{Z_m \times Z_m}$ is an element of $\tau_{Z_m \times Z_m}$
iii. The intersection of any finite member of $\tau_{Z_m \times Z_m}$ is also an element of $\tau_{Z_m \times Z_m}$

A set Z_m together with the topology $\tau_{Z_m \times Z_m}$ that is $(Z_m, \tau_{Z_m \times Z_m})$ is a topological space [12-16].

An illustration of Definition IV.I is as follows; a non-near-linear geometry with variables in Z_m where m is a non-prime integer forms a topological space as follows: Let Z_m represents the ring of integer modulo m. The geometric combination $G_m = Z_m \times Z_m$ is a collections of all subset of Z_m. $(Z_m, \tau_{Z_m \times Z_m})$ forms a topological space with the collection of its subspace $\tau_{Z_m \times Z_m}$ as topology.

A decomposition of non-prime integer as products of its prime factors is similar in analogy in this work to topological space of finite geometry. A finite intersection of lines in the geometry is analogous to finding the GCD of two or more integers while the union lines of a finite geometry is related to finding the LCM of two or more integers. The result of each is a member of the topological space in this context.

Proposition IV. III: Let m be a non-prime. The whole set $G_{\frac{m}{m_1}}, m_1 = 1$, is a member of $\tau_{Z_m \times Z_m}$.

Proof : Let $G_m = Z_m \times Z_m$, where $m \neq prime$, the following conditions hold:

(i) Since m is a non-prime, Z_m is a ring of integer modulo m. In this case, not all non-zero elements are invertible. Hence $L(au,av) \subseteq L(bu,bv), a,b,u,v \in Z_m$ and $L(au,av), L(bu,bv) \in \tau_{Z_m \times Z_m}$

(ii) Intersection of $L_1(bu,bv)$ and $L_2(bu,bv)$ gives $L_k(u,v) \in \tau_{G_d}$ where $L_k(u,v)$ is a line with at least one point.

(iii) $L_1(bu,bv) \cup L_2(bu,bv) \cup \ldots \cup L_k(bu,bv) = G_m \in \tau_{Z_m \times Z_m}$

The set $\{G_m\}$ forms a discrete topological space with the set of its subspace as topology.

V. EXAMPLES

Topology and topological space in finite Geometry $(Z_m \times Z_m, \tau_{Z_m \times Z_m})$ is verified as follows:

(i) Collection of subset $\tau_{Z_m \times Z_m}$ is a topology since G_{m_i} (for $i=1$) and G_m, $m \neq prime$ are members of the topology as G_1 is a line with one point and G_m has lines with $m_1, m_2, \ldots, m_k$ points for m_i / m.

(ii) $G_1 \cup G_2 \cup \ldots \cup G_k \in \tau_{Z_m \times Z_m}$

(iii) $G_1 \cap G_2 \cap \ldots \cap G_k \in \tau_{Z_m \times Z_m}$

Where $m_1, m_2, \ldots, m_k$ are non-trivial divisors of m.

In addition, it was observed that for $m=6$, G_6 forms a discrete topology with the set of integer modulo 6 with its subgeometries as topology. This is discussed thus; in G_6, there are lines with 6, 3, 2, and 1 points. A union of lines with 6 points is a topology in the space. An intersection of any two arbitrary lines with the same number of points gives another line whose representation is the topological space. This is shown explicitly in the following numerical examples.

Lines with 6 Points

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L(0,1) ≅ L(0,5)={(0,0),(0,1),(0,2),(0,3),(0,4),(0,5)}

L(1,0) ≅ L(5,0)={(0,0),(1,0),(2,0),(3,0),(4,0),(5,0)}

L(1,1) ≅ L(5,5)={(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)}

L(1,2) ≅ L(5,4)={(0,0),(1,2),(2,4),(3,0),(4,2),(5,4)}

L(1,3) ≅ L(5,3)={(0,0),(1,3),(2,0),(3,3),(4,0),(5,3)}

L(1,4) ≅ L(5,2) ={(0,0),(1,4),(2,2),(3,0),(4,4),(5,2)}

L(1,5) ≅ L(5,1) ={(0,0),(1,5),(2,4),(3,3),(4,2),(5,1)}

L(2,1) ≅ L(4,5) ={(0,0),(2,1),(4,2),(0,3),(2,4),(4,5)}

L(2,3) ≅ L(4,3) ={(0,0),(2,3),(4,0),(0,3),(2,0),(4,3)}

L(2,5) ≅ L(4,1) ={(0,0),(2,5),(4,4),(0,3),(2,2),(4,1)}

L(3,1) ≅ L(3,5) ={(0,0),(3,1),(0,2),(3,3),(0,4),(3,5)}

L(3,2) ≅ L(3,4) ={(0,0),(3,2),(0,4),(3,0),(0,2),(3,4)}

Lines with 3 Points

L(0,2) ≅ L(0,4)={(0,0),(0,2),(0,4)}

L(2,0) ≅ L(4,0) ={(0,0),(2,0),(4,0)}

L(2,2) ≅ L(4,4) ={(0,0)(2,2)(4,4) }

L(2,4) ≅ L(4,2) ={(0,0),(2,4),(4,2)}

Lines with 2 Points

L(0,3)={(0,0),(0,3)}

L(3,0) ={(0,0),(3,0)}

L(3,3) ={(0,0),(3,3)}

Lines with 1 Point

L(0,0) ={(0,0)}

We confirm definition IV.1 using the following as a check

Condition 1:

ϕ, $X \in \tau$, here X=Z_6 , τ=$Z_6 \times Z_6$.

The empty set ϕ is an element of the topology. That is $\phi = L(0,0) \in \tau_{Z_6 \times Z_6}$

The whole set is an element of the topology. That is , $X \in \tau_{Z_6 \times Z_6}$.

Hence, condition 1 is satisfied.

Condition 2:

Finite union of subset of $Z_6 \times Z_6$ is also an element of $Z_6 \times Z_6$. That is;

i. $L(1,1) \cup L(2,5) \cup L(3,2)=\{L(1,1), L(2,5), L(3,2)\} \in \tau_{Z_6 \times Z_6}$
Hence Condition 2 is satisfied.

Condition 3

Finite intersection of elements of $Z_6 \times Z_6$ is again an element of $Z_6 \times Z_6$

Clearly the finite intersection of elements of $Z_6 \times Z_6$ is an element of $Z_6 \times Z_6$.

i. $L(1,1) \cap L(2,5)=\{(0,0), (2,2), (4,4)\} \in \tau_{G_d}$
Hence condition 3 is also satisfied.

Thus, we conclude that at point (0,0), the geometric combination $Z_6 \times Z_6$ is a topology, and the combination $(Z_6, Z_6 \times Z_6)$ forms a topological space.

A graphical representation of lines in finite geometry G_6 and its subgeometries G_2 and G_3 is shown in figure 1 below.

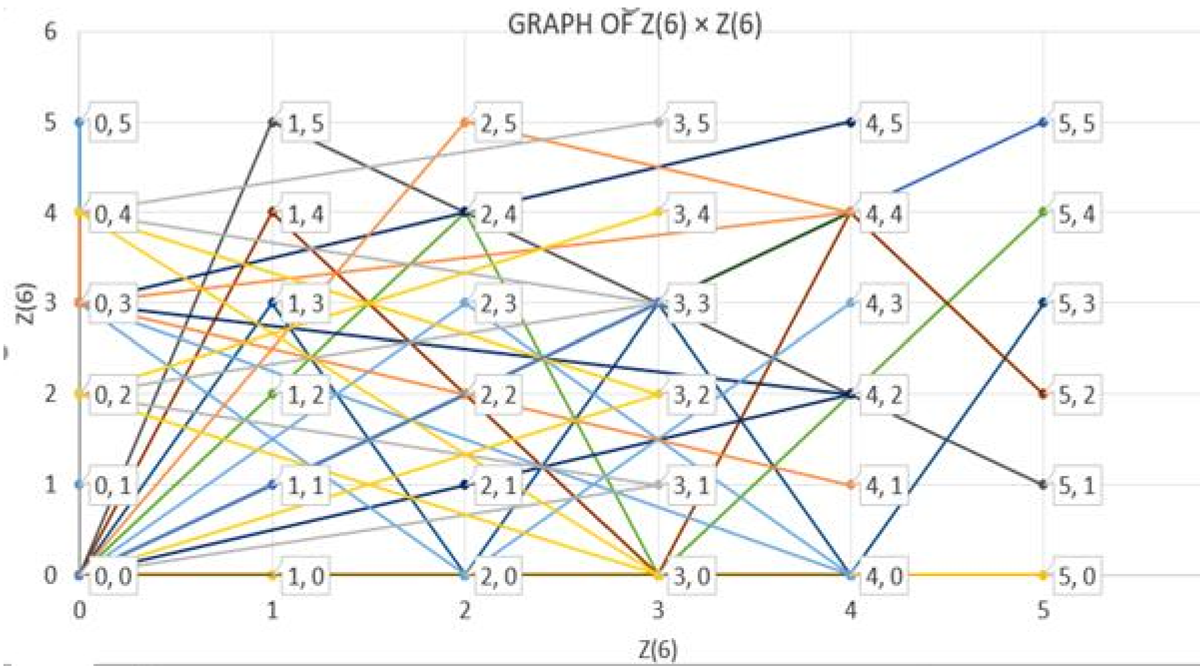

Figure 1: The Graph of finite geometry G_6.

Figure 1 above is the graph of $G_6 = Z_6 \times Z_6$. It comprises of the following $\psi(6)=12$ lines with 6 points, that is each. L(0,1), L(1,0),L(1,1),L(1,2), L(1,3), L(1,4),L(1,5), L(2,1), L(2,3),L(2,5),L(3,1) and L(3,2). $\psi(3)$ lines with 3 points, $\psi(2)$ lines with 2 points, and 1 line with 1 point (0,0). The $\psi(6)$ lines intersect at the point (0,0).

(i) $L(0,1) \cup L(1,0) \cup L(1,1) \cup L(1,2) \cup L(1,3) \cup L(1,4) \cup L(1,5) \cup L(2,1) \cup L(2,3) \cup L(2,5) \cup L(3,1) \cup L(3,2) = \{L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4), L(1,5), L(2,1), L(2,3), L(2,5), L(3,1), L(3,2)\} \in G_6$

(ii) $L(1,2) \cap L(1,5)=\{L(0,0), L(2,4), L(4,2)\} \in G_6$

(iii) $G_6 \preccurlyeq G_6$

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(iv) $G_1=\{L(0,0)\}\in G_6$

The union of the lines in G_6 contains lines with 6,3, and 2 points. These lines are members of the space. The intersection of any pairs of arbitrary lines with the same number of points in the geometry G_6 gives a line $L(\vartheta,\sigma)$, which is a member of the topological space.

Hence, G_6 forms a discrete topological space with its subgeometries G_1, G_2, $and\, G_3$ as topology.

CONCLUSION

A study on topological space which exists in non-near-linear finite geometry was carried out in this work. Two or more lines in non-near-linear finite geometry intersect in at least one point in the geometry. These points are members of the topological space with the intersecting points as topology. A union of lines in the geometry is a member of the topology. An existence of non-near-linear finite geometry which originated from non-prime dimensional finite geometry leads to a discrete topology. More importantly, it was discovered in this work that, divisor function played a key role in the formation of discrete topological space in finite geometry via partial ordered relation. The idea presented in this work has potential of being used for coding of message in information theory. For instance, a code in the context of this work is a finite geometry G_d over Z_d. It can be a set of messages stored in a medium such as magnetic disk over a period of time. To verify whether or not the message has been tampered with, parties involved will need to check for a bijection between the content of what was stored and what is found later in the storage medium.

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