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Topological Space: A Space in Finite Dimensional Euclidean Plane with Variables in $Z(q)$

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ABSTRACT

This work centres on a finite dimensional Euclidean geometry with variables in Z(q) called near-linear finite geometry. In it, we investigate points and lines in the geometry. A set of points joined together forms lines in the geometry and a presence of trivial divisors yields a set together with the collection of only two subsets, that is, the whole set itself and integer one as members with subgeometries as induced topology where the only open set is the universal set and an empty set. We prove that the Euclidean plane Z(q)× Z(q) for prime q forms an indiscrete topological space whose open sets are trivial subgeometries, $G(1)$ *and* $G(q)$.

Keywords: Topology, topological space, indiscrete topology, field of integer modulo q, lines, Near-linear finite geometry.

INTRODUCTION

In the context of modular arithmetic, Euclidean geometry involves working with coordinates and geometric concepts within a finite set of integers by using the modulo operator. In this work, the notation, $Z(q)$ denotes an integer modulo q. The Cartesian products $Z(q)\times Z(q)$ gives a finite geometry $G(q)$. The geometry is defined as the pair $(L(q), P(q))$ where $L(q)$ denotes the set of lines in a geometry and $P(q)$ denotes the set of points in a line. A lot of work has been ongoing focusing on finite geometry where, for instance, two lines in a finite geometry intersect mostly at a point. The findings in [1-2] negated this common axiom by restricting its validity to lines in near-linear geometry. It was affirmed in [3] that, in non-near-linear finite geometry, two lines of the geometry intersect in at least one point.

Researchers in topology have in the past been confronted challenges focusing on the application of topology outside classrooms and the relevance of the area to human endeavour. For instance, authors [4-6] have asked a pertinent question about the areas of application or relevance of topology to humanity outside teaching. In what can be described as an answer to those questions, Mayila et al. [7] developed a mathematical representation of a decision space and a topology on a nation using some properties of topological operators. This further underscores the importance of topology and the need to inquire more into its possible areas of real-life applications.

In this work, we investigate the structure of topology on finite geometry. We examine how topological structure is established on near-linear finite geometry. We demonstrate how finite geometry under consideration satisfies conditions of topology. This work is partitioned into seven sections as follows. Section I covers the introduction. Section II focuses on terminologies used in this work. It is titled preliminaries. Lines in Euclidean plane with variables in Z(q) is discussed in section III. Section IV focuses on lines in Euclidean geometry $Z(q)\times Z(q)$. Topological space in finite geometry $Z(q)\times Z(q)$ is discussed in section V of this work. In section

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VI, we showcase illustration of this work using numerical examples. Section VII concludes this work.

PRELIMINARIES

Definitions II.1

i. The field of integer modulo q is denoted by $Z(q)$.

ii. In this work, the notation $G(q)$ represents Euclidean plane with q points in each line of the plane.

iii. The cardinality of $Z(q)$ is $\phi(q)$ that is, Euler Phi function where

$$\phi(q)=q\prod_{k=1}^{n}\left(1-\frac{1}{q_k}\right),\ \text{in this work } q_k \text{ is a prime integer.} \quad (1)$$

iv $\psi(q)=q\prod_{k=1}^{n}\left(1+\frac{1}{q_k}\right)$ denotes Dedekind psi function where;

(2)

v. $GCD(a,b)$ represents the Greatest Common Divisor of integer a and b, where $a,b \in Z(q)$.

LINES IN EUCLIDEAN PLANE WITH VARIABLES IN $Z(q)$

In this section, lines in near-linear finite geometry are discussed. This concept forms the theoretical background which this work is laid on. The concepts are defined in [1-2] as follows:

Definition III. 1: A space $S(P,\mathrm{L})$ is a system of points P and line L such that every line L is a subset of P.

Definition III. 2: A near-linear space is an incident structure $I(P,\mathrm{L})$ of points P and lines L such that the following axioms are satisfied:

i. Any line has at least two points.

ii. Two lines meet in at most one point.

A near-linear space is defined as follows:

$$G(q)=(\mathrm{L}(q),P(q)) \quad (3)$$

Here, $P(q)$ represents points on the line $\mathrm{L}(q)$.

$\mathrm{L}(q)$ denotes lines with point $P(q)$, where

$$\mathrm{L}(q)=\{(\alpha a,\alpha b)\,|\,a,b\in Z(q)\},\ \alpha\in Z(q) \quad (4)$$

Lemma III.3: Let q be a prime. Two distinct lines of a near-linear finite geometry $G(q)$ intersect at one point.

Proof:

Let $G(q)=Z^2(q)$, where

$Z^2(q)= Z(q)\times Z(q)$ represents lines with points in $G(q)$. For q a prime, intersections of any pair of arbitrary lines yields a point.

LINES IN EUCLIDEAN GEOMETRY $Z(q)\times Z(q)$

This subsection focuses on lines in near-linear finite geometry. Here two lines in meet at one point, that is the origin. Equation (4) denotes a line through the origin (0,0).

Attention of this work centers on lines at any arbitrary point amongst the points under consideration. Based on the definition of [8], lines in $G(q)$ is defined as follows:

$$L(q)=\{\alpha a+\vartheta,\alpha b+s|a, b, \vartheta, s\in Z(q)\}, \alpha\in Z(q) \quad (5)$$

Mathematically, it is defined as the pair $(P(q),L(q))$ in $G(q)=Z^2(q)$. Here,

$P(q)$ represents points in a line and $L(q)$ represents lines in $G(q)$ where,

$$P(q)=\{(s,t)|s,t\in Z(q)\} \quad (6)$$

In this work, literatures of [3], [9-10] were consulted and the following propositions were confirmed in their works.

1. If $a\in Z(q)$ is an invertible element, then $L(\rho,\vartheta)=L(a\rho,a\vartheta)$.
Else, $L(\rho,\vartheta)\ mod(q)\subset L(a\rho,a\vartheta)$, and $L(a\rho,a\vartheta)\prec L(\rho,\vartheta)$, where $\prec$ represents partial ordering.
We confirm that $L(\rho,\vartheta)$ is a maximal line in $G(q)$ if $GCD(\rho,\vartheta)\in Z^*(q)$ and $L(\rho,\vartheta)$ is a subline in $G(q)$ if $GCD(\rho,\vartheta)\in Z(q)-Z^*(q)$
2. A Euclidean plane $G(q)$ shown in equation (5) can also be defined as
$$L(rx,ry)=\{(rtx,rty)|x, y\in Z(q)\}, r\in Z(qr) \quad (7)$$
In addition, the line $L(rx,ry)$ in $G(q)$ is a subline of
$$L(x,y)=\{(t'x,t'y)|\ t'=0,...,rd-1\}$$
3. For $q|d$, if two maximal lines have k points in common, then the k points gives a subline $L(x,y)$ where $x,y\in\frac{q}{d}Z(d)$.

If we consider the subgeometry $G(d)$, the subline $L(x,y)$ in $G(q)$ is a maximal line in $G(d)$. An existence of $\psi(d)$ maximal lines is confirmed in sub-geometry $G(d)$ of finite geometry $G(q)$. A set of lines in non-near-linear finite geometry $G(q)$ together with its set of subgeometry forms a topological space via partial ordering with its subgeometry as embedded in the space.

TOPOLOGICAL SPACE IN $Z(q)\times Z(q)$

In this subsection, we discuss in detail the derivation of Euclidean plane from $Z(q)\times Z(q)$ where all set of points derived from the geometry form a topological space, with their subsets as topology. For q a prime, $Z(q)\times Z(q)$ form a near-linear finite geometry. Lines in the geometry were generated. The union of arbitrary lines is a member of the topological space. The intersection of any arbitrary finite lines gives a line with one point in this case. The set of all the lines in $Z(q)\times Z(q)$, the lone line with one point in the geometry under discourse forms an indiscrete topological space with its subspace $G(1)$ and $G(q)$ as topology.

Here, we present how a phase-space finite geometry forms a topological space with its subset as topology.

Definition V.I: A pair (X, τ) where X denotes a nonempty set and τ denotes collection of subsets. (X, τ) is called a topological space if it meets the following conditions:

i. The empty set and the whole set are elements of τ, that is, $\varphi, X\in\tau$.

ii. The union of any member of τ is also an element of τ.
iii. The intersection of any finite member of τ is also an element of τ.

Hence, X together with the collection of its subsets τ, that is (X, τ) form a topological space [11-16].

In this work, a near-linear geometry with variables in $Z(q)$ where q is a prime integer forms a topological space as follows: suppose X denotes a field of integer modulo q, let $X=Z(q)$. Then the collections of all subsets of $Z(q)\times Z(q)$ form a topology. As a result, $(Z(q), Z(q)\times Z(q))$ forms a topological space. This phenomenon is shown to exist when the geometric lines are taken through any arbitrary points in the geometry as defined in equation (5). In general, for d a non-trivial divisor of q, any subsets of $\{Z(q)\times Z(q)\}$ is an open set and the elements of $Z(q)\times Z(q)$ formed open sets. A finite intersection of subsets of the collection $Z(q)\times Z(q)$ yields an open set. The set X and $\{\ \}$ of $Z(q)\times Z(q)$ are both open and closed. However, in this context since q is a prime, the set $G(q)$ together with the collection of its subsets $\tau_{G(q)}$ form an indiscrete topological space with its trivial divisor, $Z(1)\times Z(1)$ and $Z(q)\times Z(q)$ as induced topology.

Suppose $x, y \in Z(q)$, we define $M_{x,y}=\{x+ym \mid m\in Z(q)\}$. A non-empty subset $S\subseteq Z(q)$ is open if it is a union of sets of the form $M_{x,y}$. The collection of subsets obtained from $M_{x,y}$ forms a topology. Furthermore, if we take a finite intersection of any finite subsets of the whole collection. It gives an open set.

Let X be the field of integer modulo q and $G(q)=Z(q)\times Z(q)$ be the collection of all subsets of $Z(q)$. Then $(G(q), \tau_{G(q)})$ forms a topological space with the subset of $G(q)$ as topology.

As an illustration in this context, a topological space in finite Euclidean plane is analogous to expressing an integer as products of its prime. Existence of trivial subgeometry in the Euclidean is related to expression of prime integer as products of itself and integer one, considering the fact that prime numbers has only two factors, that is itself and integer one. A finite intersection of lines in the geometry is related to finding the greatest or highest common factor (H.C.F) of two integers while the union of two or more subgeometries of a finite Euclidean plane in this context is related to finding the least common multiplier (L.C.M) of two or more integers. The result of each is a member of the topological space.

Proposition V. IV: Let q be a prime. The empty set $G_{\frac{q}{q}}$ is a member of τ.

Proof : Let $G(q)=Z(q)\times Z(q)$, where $q=prime$, the following conditions hold:

Since q is a prime then $Z(q)$ is a field of integer modulo q. All the non-zero elements in this regard are invertible. Hence $L(\rho,\vartheta)\cong L(a\rho,a\vartheta)$, $\rho, a, \vartheta \in Z(q)$, $L(a\rho,a\vartheta)\in \tau_{G(q)}$

(i) Intersection of $L_1(a\rho,a\vartheta)$ and $L_2(a\rho,a\vartheta)$ gives $L_k(a,\vartheta)\in\tau_{G(q)}$ where $L_k(a,\vartheta)$ is a line with at most one point.

(ii) $L_1(a\rho,a\vartheta)\cup L_2(a\rho,a\vartheta)\cup\ldots\cup L_k(a\rho,a\vartheta)=G(q)\in\tau_{G(q)}$

Hence forms an indiscrete topological space.

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EXAMPLES

In this section, we demonstrate how a topological space is established on near-linear finite geometry using numerical examples. The set $\{G(q)\}$, together with the collection $\tau_{G(q)}$ form a topological space as follows:

(i) Collection of subset $\tau_{G(q)}$ is a topology since $G(q)$ (for i=1) and $G(q)$, $q=prime$ are members of the topology as $G(1)$ is a line with one point and $\{G(q)\}$ has lines with q points.

(ii) $G(1) \cup G(2) \cup ... \cup G(k) \in \tau_{G(q)}$

(iii) $G(1) \cap G(2) \cap ... \cap G(k) \in \tau_{G(q)}$

Where 1, and q are trivial divisors of q

In addition, it was confirmed that for q=5, $G(5)$ forms an indiscrete topological space with its subgeometries $G(1)$ and $G(5)$ topology. The result is discussed as follows; for $G(5)$, there are 6 lines 5 points each. The union of lines with 5 points is a member of the topological space. An intersection of any two arbitrary lines in the geometry gives point (0,0) which is a member of the space. Hence, $G(1)$ and $G(5)$ are topologies and the set$\{G(5), G(1)\}$ forms a topological space with its subspace $G(1)$ and $G(5)$ as topology. Furthermore, each member of the space is an open set.

The following results were generated from equation (3)

$$L(0,1) \cong L(0,2) \cong L(0,3) \cong L(0,4) \quad (13)$$

$$L(1,0) \cong L(2,0) \cong L(3,0) \cong L(4,0) \quad (14)$$

$$L(1,1) \cong L(2,2) \cong L(3,3) \cong L(4,4) \quad (15)$$

$$L(1,2) \cong L(2,4) \cong L(3,1) \cong L(4,3) \quad (16)$$

$$L(1,3) \cong L(2,1) \cong L(3,4) \cong L(4,2) \quad (17)$$

$$L(1,4) \cong L(2,1) \cong L(3,4) \cong L(4,2) \quad (18)$$

As an illustration,

L (1,3) about the origin in the geometry has the following set of points, {(0,0),(1,3)(2,1)(3,4)(4,2)}.

Figure 1 below shows a diagrammatic representation of the Euclidean plane geometry $G(5)$.

It comprises of the following $\psi(5)$ lines, that is L (0,1), L (1,0), L (1,1), L (1,2), L (1,3), and L (1,4). The $\psi(5)$ lines intersect at the point (0,0).

(i) $L(0,1) \cup L(1,0) \cup L(1,1) \cup L(1,2) \cup L(1,3) \cup L(1,4) = \{L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4)\} \in G(5)$

(ii) $L(0,1) \cap L(1,0) \cap L(1,1) \cap L(1,2) \cap L(1,3) \cap L(1,4) = L(0,0)$

The union of the lines in $G(5)$ is a member of the space. The intersection of the lines in $G(5)$ gives a line with point (0,0), which is a member of the topological space.

Hence, $G(5)$ forms an indiscrete topological space with $G_5 = \{L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4)\}$ and $G(1) = \{L(0,0)\}$ as topology.

We confirm definition V.1 using the following as a check

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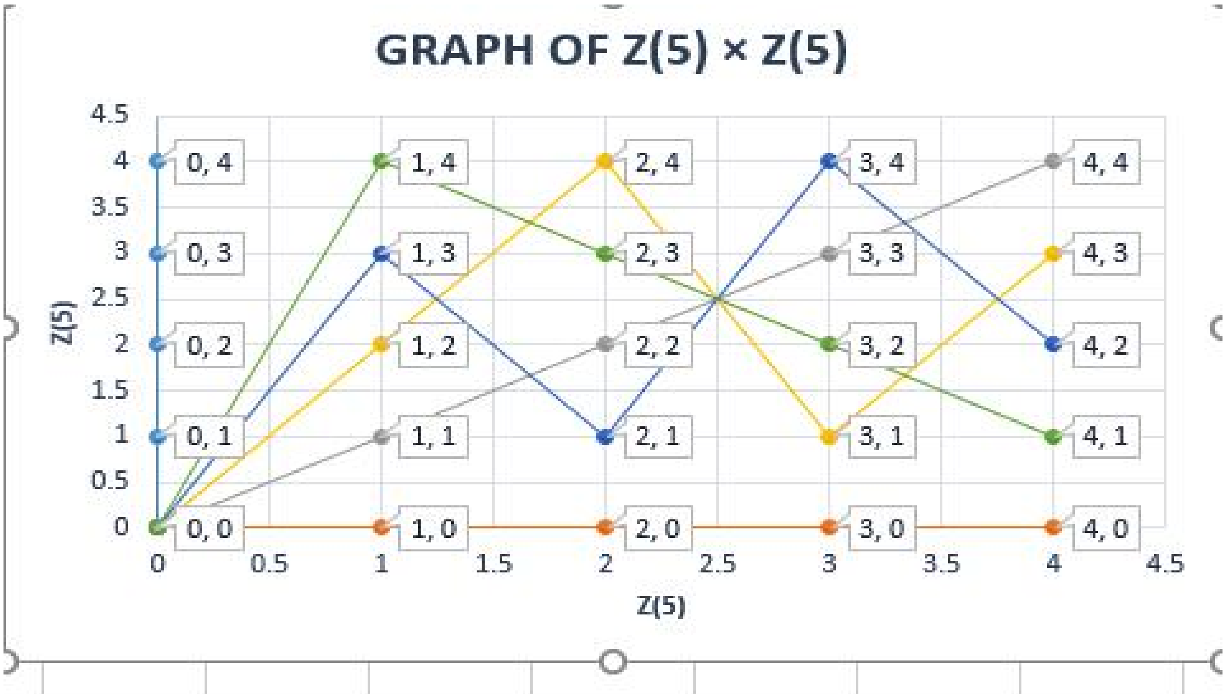

Figure 1: The Graph of $G_5 = Z_5 \times Z_5$.

Condition 1:

$\phi, X \in \tau$, here $X = Z(5)$, $\tau = Z(5) \times Z(5)$.

ϕ is an element of the topology. In this work, $\phi = \mathrm{L}(0,0) \in \tau_{G(q)}$

The whole set X is an element of $\in \tau_{G(q)}$.

Hence condition 1 is satisfied.

Condition 2:

A union of subset of $Z(5) \times Z(5)$ is an element of $Z(5) \times Z(5)$.

That is;

i. $\mathrm{L}(1,1) \cup \mathrm{L}(2,3) \cup \mathrm{L}(3,2) = \{\mathrm{L}(1,1), \mathrm{L}(2,3), \mathrm{L}(3,2)\} \in \tau_{G(5)}$

Hence condition 2 is satisfied.

Condition 3

A finite intersection of elements of $Z(5) \times Z(5)$ is again an element of $Z(5) \times Z(5)$.

i. $\mathrm{L}(3,4) \cap \mathrm{L}(2,0) = \{(0,0)\} \in \tau_{G(5)}$

Hence condition 3 is also satisfied.

Thus, we conclude that at point (0,0), the geometric combination $Z(5) \times Z(5)$ is a topology, and the combination $(Z(5), Z(5) \times Z(5))$ forms a topological space.

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CONCLUSION

This work investigated topological space which exists in a finite dimensional Euclidean plane with variables in integer modulo q. It was discovered that two or more lines, a member of topology in the geometry meet at one point and the point is a member of the topology. A union of lines in the geometry is a member of the topology. An existence of near linear finite geometry which originated from prime dimensional finite geometry leads to an indiscrete topology. More importantly, an existence of two factors in a prime integer leads to an indiscrete topological space with its trivial subgeometry as topology.

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